# Eigenvalue Reanalysis of Locally Modified Structures Using a Generalized Rayleigh's Method

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**Approximate eigenvalue reanalysis methods for locally modified structures are developed based on the generalized Rayleigh's quotients. For simple modifications such as adding springs and masses or changing the truss member's cross-sectional area, closed-form formulas are included. Several applications are presented.**

## Nomenclature

| | |
|---|---|
| $[a]$ | $= [k]^{-1}$ |
| $\{a_J\}$ | $=J$th column of $[a]$ |
| $[B]$ | = Boolean matrix |
| $C_x, C_y, C_z$ | = direction cosines of modified truss member (in Table 1) |
| $[k]$ | = original system stiffness matrix |
| $[\Delta k]$ | $= [B]^T [\hat{\Delta} k] [B]$ |
| $[m]$ | = original system mass matrix |
| $M_{ii}$ | $= i$-ith component of the mass matrix $[m]$ |
| $\underset{t\times t}{[\Delta m]}$ | $= \underset{t\times n}{[B]^T} \underset{n\times n}{[\Delta m]} \underset{n\times t}{[B]}$ |
| $[\Delta m], [\Delta k]$ | = modifications in mass stiffness matrix |
| $[\hat{\Delta} m], [\hat{\Delta} k]$ | = nonzero submatrices of $[\Delta m]$ and $[\Delta k]$ |
| $n$ | = number of degrees of freedom in the system |
| $t$ | = number of degrees of freedom in $\{\phi\}$ |
| $\{x^{(i)}\}$ | $= i$th improved trial vector |
| $\{x^{(0)}\}$ | = initial trial vector |
| $[Y]$ | $= [K]^{-1} [B]^T$ |
| $\lambda$ | = original system fundamental eigenvalue |
| $\lambda^{(1)}$ | = inverse iteration $= \lambda^{(1,1)}$ |
| $\lambda^{(i,j)}$ | = generalized Rayleigh's quotient using $\{x^{(j)}\}$ in computing the strain energy, $j=i-1$ or $i$ |
| $\lambda_R$ | = Rayleigh's quotient $= \lambda^{(0,0)}$ |
| $\lambda_T$ | = Timoshenko's quotient $= \lambda^{(0,1)}$ |
| $\{\phi\}$ | = original system first modal vector |
| $\{\hat{\phi}\}$ | = components of $\{\phi\}$ that are connected to the modification $= \underset{t\times n}{[B]} \underset{n\times 1}{\{\phi\}}$ |

## Introduction

EIGENVALUE reanalyses of locally modified discrete systems have been studied by many investigators.[1-28] Both approximate[3-18] and exact reanalysis methods[19-24] have been reported in the literature, which are reviewed in Ref. 1. The applicability of the methods are patently different. For example, the approximate method cannot treat a large variation of parameters without incurring poor accuracy. On the other hand, exact methods can treat large parameter variations, although this involves the solution of a nonlinear eigenvalue problem.

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In this paper, we introduce approximate reanalysis methods based on the generalized Rayleigh's quotient. With these methods, the fundamental eigenvalue of a locally modified system can be computed readily, using the fundamental mode as well as the static solution of the original system. Following a general development of theory, several special cases of interest in design are derived here and then tabulated. Numerical examples illustrate the accuracy of the new formulations.

## Generalized Rayleigh's Quotient

Rayleigh's method for computing the eigenvalues of a conservative system utilizes an assumed mode for sinusoidal motion and then equates the maximum kinetic energy to the maximum potential (strain) energy. That is,

$$T_{\max} = V_{\max} \tag{1}$$

For a discrete system described by a stiffness matrix $[k]$, mass matrix $[m]$, and an assumed mode shape of $\{x^{(0)}\}$,

$$T_{\max} = \tfrac{1}{2}\omega^2 \{x^{(0)}\}^T [m] \{x^{(0)}\} \tag{2a}$$

$$V_{\max} = \tfrac{1}{2} \{x^{(0)}\}^T [k] \{x^{(0)}\} \tag{2b}$$

Substitution of Eqs. (2) into Eq. (1) leads to the well-known Rayleigh's quotient

$$\omega^2 = \lambda = \frac{\{x^{(0)}\}^T [k] \{x^{(0)}\}}{\{x^{(0)}\}^T [m] \{x^{(0)}\}} \tag{3}$$

In the generalized Rayleigh's quotient, improved assumed modes $\{x^{(i)}\}$ are used in the evaluation of kinetic and potential energy. For example, with $\{x^{(0)}\}$ given, $\{x^{(1)}\}$ can be computed by solving the equation

$$[k]\{x^{(1)}\} = \lambda [m] \{x^{(0)}\} = \{f^{(1)}\} \tag{4}$$

The potential energy corresponding to the assumed mode $\{x^{(0)}\}$ can be computed as

$$\begin{aligned} V^{(1)}_{\max} &= \tfrac{1}{2} \{x^{(1)}\}^T [k] \{x^{(1)}\} \\ &= \tfrac{1}{2} \{x^{(1)}\}^T \lambda [m] \{x^{(0)}\} \end{aligned} \tag{5}$$

From Eq. (4),

$$\{x^{(1)}\} = \lambda [k]^{-1} [m] \{x^{(0)}\} \tag{6}$$

Substitution of Eq. (6) into Eq. (5) permits $V_{max}^{(1)}$ to be expressed in terms of the original assumed mode $\{x^{(0)}\}$,

$$(\lambda^2/2)\{x^{(0)}\}^T[m][k]^{-1}[m]\{x^{(0)}\} \tag{7}$$

Use the original assumed mode in calculating the maximum kinetic energy. Equate Eqs. (2a) and (7), so that

$$\lambda^{(0,1)} = \frac{\{x^{(0)}\}^T[m]\{x^{(0)}\}}{\{x^{(0)}\}^T[m][k]^{-1}[m]\{x^{(0)}\}} \tag{8}$$

In Eq. (8), the superscripts (0,1) indicate that we use $\{x^{(0)}\}$ to compute kinetic energy and $\{x^{(1)}\}$ to compute the potential energy. This is the discrete version of the $R_{01}$ method described by Clough and Penzien.[17]

The method can be generalized to the following algorithm:

1) With $\{x^{(i)}\}$ known, compute $\{x^{(i+1)}\}$ from

$$[k]\{x^{(i+1)}\} = \lambda[m]\{x^{(i)}\} \tag{9}$$

2) The method $R_{i,i+1}$, which leads to $\lambda^{(i,i+1)}$, proceeds in the following way. Compute

$$T_{max} = ½\lambda\{x^{(i)}\}^T[m]\{x^{(i)}\} \tag{10a}$$

$$V_{max} = ½\{x^{(i+1)}\}^T[k]\{x^{(i+1)}\} \tag{10b}$$

Equate $T_{max} = V_{max}$ and compute $\lambda^{(i,i+1)}$.

3) Method $R_{i+1,i+1}$, which leads to $\lambda^{(i+1,i-1)}$, uses

$$T_{max} = ½\lambda\{x^{(i+1)}\}[m]\{x^{(i+1)}\} \tag{11}$$

and Eq. (10b) for $V_{max}$.

It should be noted that method $R_{i,i}$ is the same as the well-known inverse iteration procedure in its $i$th iteration.

## Timoshenko's Quotient as a Generalized Rayleigh's Quotient

Timoshenko's quotient $\lambda_T$ given in Ku[29] is

$$\lambda_T = V^TBV/V^TCV \quad \text{and} \quad C = BA^{-1}B \tag{12}$$

In terms of the present nomenclature,

$$v = \{x^{(0)}\}, \quad [B] = [m], \quad [A] = [k]$$

$$[C] = [m][k]^{-1}[m]$$

and

$$\lambda_T = \frac{\{x^{(0)}\}^T[m]\{x^{(0)}\}}{\{x^{(0)}\}^T[m][k]^{-1}[m]\{x^{(0)}\}} \tag{13}$$

Thus, $\lambda_T = \lambda^{(0,1)}$.

## Inverse Iteration and the Generalized Rayleigh's Quotient

In inverse iteration, the eigenvalue at the $r$th iteration is computed from

$$\lambda^{(r)} = \frac{\{x^{(r)}\}^T[k]\{x^{(r)}\}}{\{x^{(r)}\}^T[m]\{x^{(r)}\}} \tag{14a}$$

with

$$[k]\{x^{(r)}\} = [m]\{x^{(r-1)}\} \tag{14b}$$

Specifically, for $r=1$,

$$\lambda^{(1)} = \frac{\{x^{(1)}\}^T[k]\{x^{(1)}\}}{\{x^{(1)}\}^T[m]\{x^{(1)}\}} \tag{15a}$$

and

$$\{x^{(1)}\} = [k]^{-1}[m]\{x^{(0)}\} \tag{15b}$$

Substitute Eq. (15b) into Eq. (15a)

$$\lambda^{(1)} = \frac{\{x^{(0)}\}^T[m][k]^{-1}[m]\{x^{(0)}\}}{\{x^{(0)}\}^T[m][k]^{-1}[m][k]^{-1}[m]\{x^{(0)}\}} \tag{16}$$

To evaluate $\lambda^{(1,1)}$, $V_{max}$ is given by Eq. (7) and $T_{max}$ should by computed from $\{x^{(1)}\}$. That is,

$$T_{max} = ½\lambda\{x^{(1)}\}^T[m]\{x^{(1)}\} \tag{17}$$

Substitute $\{x^{(1)}\}$ from Eq. (6) into Eq. (17) to get

$$T_{max} = ½\lambda^3\{x^{(0)}\}^T[m][k]^{-1}[m][k]^{-1}[m]\{x^{(0)}\} \tag{18}$$

Equating Eqs. (7) and (18)

$$\lambda^{(1,1)} = \frac{\{x^{(0)}\}^T[m][k]^{-1}[m]\{x^{(0)}\}}{\{x^{(0)}\}^T[m][k]^{-1}[m][k]^{-1}[m]\{x^{(0)}\}} \tag{19}$$

By comparison of Eqs. (16) and (19), it can be concluded that

$$\lambda^{(1)} = \lambda^{(1,1)}$$

In general, $\lambda^{(r)} = \lambda^{(r,r)}$, the eigenvalue obtained by inverse iteration after $r$ iterations.

## Eigenvalue Reanalysis Using the Generalized Rayleigh's Quotient

When a system is modified locally, it is often reasonable to assume that the mode shape does not change drastically. Let $[m]$, $[k]$ be the stiffness and mass matrices of an $n$ degree-of-freedom (dof) system, with known fundamental eigenvalue $\lambda$ and corresponding mode shape $\{\phi\}$. Suppose this system is modified by $[\Delta m]$ and $[\Delta k]$ and an approximation to the new eigenvalue $\lambda$ is sought. To this end, we choose to use Rayleigh's quotient, Timoshenko's quotient, and inverse iteration.

Assume $\{x^{(0)}\} = \{\phi\}$ with $\{\phi\}^T[m]\{\phi\} = k$. Then Rayleigh's quotient becomes

$$\lambda_R = \frac{\phi^T(k+\Delta k)\phi}{\phi^T(m+\Delta m)\phi}$$

or

$$\lambda_R = \frac{\lambda + \phi^T\Delta k\phi}{1+\phi^T\Delta m\phi} \tag{20}$$

If $[\hat{\Delta k}]$ and $[\hat{\Delta m}]$ are nonzero submatrices of $[\Delta k]$ and $[\Delta m]$ then Eq. (20) becomes

$$\lambda_R = \frac{\lambda + \{\hat{\phi}\}^T[\Delta\hat{k}]\{\hat{\phi}\}}{1+\{\hat{\phi}\}^T[\hat{\Delta m}]\{\hat{\phi}\}} \tag{21}$$

The Timoshenko quotient [Eq. (13)] can be written as

$$\lambda_T = \frac{\{\phi\}[(m+\Delta m)]\{\phi\}}{\{f\}^T\{g\}} \tag{22}$$

where

$$\{f\} = [m+\Delta m]\{\phi\} \tag{23}$$

and

$$[k+\Delta k]\{g\} = \{f\} \tag{24}$$

Let

$$[\Delta k] = [B]^T[\hat{\Delta k}][B] \tag{25}$$

then use static reanalysis methodology[30] to obtain

$$\{g\} = \{g_1\} - [k^{-1}B^T][\hat{\Delta k}][I+\hat{k}^{-1}\hat{\Delta k}]^{-1}\hat{g}_1 \tag{26}$$

where

$$\{g_1\} = [k]^{-1}\{f\} = [k]^{-1}([m+\Delta m])\{\phi\} \tag{27}$$

$$\{\hat{g}_1\} = [B]\{g_1\} \tag{28}$$

Now,

$$\{g_1\} = \{k^{-1}m\phi\} + \{k^{-1}\Delta m\phi\}$$

or

$$\{g_1\} = \{g_0\} + \{Y\hat{\Delta M}\hat{\phi}\} \tag{29}$$

where

$$\{g_0\} = k^{-1}m\phi = (1/\lambda)\{\phi\} \tag{30}$$

Thus, Timoshenko's quotient can be summarized by the algorithms

1) Compute

$$\{g_0\} = (1/\lambda)\{\phi\} \tag{31}$$

2) Solve $[Y]$ from

$$[k][Y] = [B]^T \tag{32}$$

3) Compute

$$\{g_1\} = \{g_0\} + [Y][\hat{\Delta m}]\{\phi\} \tag{33}$$

where

$$\{\hat{\phi}\} = [B]\{u\} \tag{34}$$

4) Compute

$$\{g\} = \{g_1\} - [Y][\hat{\Delta k}](I+\hat{k}^{-1}\hat{\Delta k})^{-1}\{\hat{g}_1\} \tag{35}$$

5) Compute Timoshenko's quotient

$$\lambda_T = \frac{1+\hat{\phi}^T\hat{\Delta m}\hat{\phi}}{\{f\}^T\{g\}} \tag{36}$$

Note that both $\lambda_R$ and $\lambda_T$ are upper bounds for the lowest $\lambda'$, the true eigenvalue of the modified system. Using the lower bound expression in Ku,[29] we get

$$\lambda_L < \lambda' < \lambda_T \tag{37}$$

where

$$\lambda_L = \lambda_R + \sqrt{\lambda_R(\lambda_R - \lambda_T)} \tag{38}$$

and $\lambda_R$ and $\lambda_T$ are given in Eqs. (20) and (36), respectively.

Note that the vector $\{g\}$ is an improved mode shape in that it is the mode shape after one step of the inverse iteration starting with the original mode shape $\{\phi\}$. Thus, the approximate eigenvalue from this new eigenvector can be computed from

$$\lambda^{(1)} = \frac{\{g\}^T[k+\Delta k]\{g\}}{\{g\}^T[m+\Delta m]\{g\}} = \frac{\{g\}^T\{f\}}{\{g\}^T[m+\Delta m]\{g\}} \tag{39}$$

Note that $\{g\}^T\{f\} = \{f\}^T\{g\}$ has been computed in the evaluation of $\lambda_T$ from Eq. (22).

With $\lambda^{(1)}$ and $\lambda_T$ available, an improved lower bound $\bar{\lambda}_L$ can be computed,

$$\bar{\lambda}_L = \lambda_T - \sqrt{\lambda_T(\lambda_T - \lambda^{(1)})} \tag{40}$$

To summarize, we have

$$\lambda_L < \bar{\lambda}_L < \lambda^{(1)} < \lambda_T < \lambda_R \tag{41}$$

## Special Case: Mass Modification Only

Suppose the modification in mass does not affect the system stiffness property, i.e., $[\Delta k] = [0]$. Equations (34) and (35) lead to the equation for the improved mode shape vector

$$\{g\} = (1/\lambda)\{\phi\} + [Y][\hat{\Delta m}]\{\hat{\phi}\} \tag{42}$$

where

$$[Y] = [k]^{-1}[B]^T \tag{43}$$

For this case, the Rayleigh's quotient becomes

$$\lambda_R = \lambda/(1+\{\hat{\phi}\}^T[\hat{\Delta m}]\{\hat{\phi}\})$$

and the Timoshenko's quotient becomes

$$\lambda_T = \frac{\lambda(1+a)}{(1+2a)+\lambda\{\phi\}^T[\Delta m][Y][\Delta m]\{\phi\}} \tag{44}$$

$$a = \{\hat{\phi}\}^T[\hat{\Delta m}]\{\hat{\phi}\} \tag{45a}$$

$$b = \{\hat{\phi}\}^T[\hat{\Delta m}][\hat{Y}][\hat{\Delta m}]\{\hat{\phi}\} \tag{45b}$$

and the formula based on inverse iteration would be

$$\lambda^{(1)} = \frac{\{g\}^T\{f\}}{\{g\}^T[m+\Delta m]\{g\}} = \frac{A}{B} \tag{46}$$

where

$$A = (1/\lambda)(1+2a+\lambda b)$$

$$B = (1/\lambda^2)[1+3a+2\lambda b+\lambda^2(c+d)]$$

and

$$c = \{\hat{\phi}\}^T[\hat{\Delta m}][Y]^T[m][Y][\hat{\Delta m}]\{\hat{\phi}\} \tag{47a}$$

$$d = \{\hat{\phi}\}^T[\hat{\Delta m}][\hat{Y}][\hat{\Delta m}][\hat{Y}][\hat{\Delta m}]\{\hat{\phi}\} \tag{47b}$$

For the special case where mass $m$ is introduced at dof $I$, we have

$$[\hat{\Delta m}] = m, \quad \{\hat{\phi}\} = \phi_I$$

$$[Y] = \{a_I\} = I\text{th column of } [k]^{-1}$$

$$[Y]^T[m][Y] = \sum_{i=1}^{n} m_{ii}a_{iI}$$

Thus,

$$a = \phi_I^2 m, \quad b = m^2\phi_I^2 a_{II}$$

$$B = \frac{1}{\lambda^2}(1+3m\phi_I^2) + \frac{1}{\lambda}(2m^2\phi_I^2 a_{II}) + m^2\phi_I^2\left(\sum_{i=1}^{n} m_{ii}a_{iI}^2\right) + m^3 a_{II}^2\phi_I^2 \tag{48a}$$

and

$$A = (1/\lambda)(1 + 2m\phi_I^2 + \lambda m^2 \phi_I^2 a_{II}) \tag{48b}$$

$\lambda^{(1)}$ can then be evaluated from Eq. (46),

$$\lambda^{(1)} = \frac{\lambda[1 + 2\phi_I^2 m + (\phi_I^2 a_{II} m^2 \lambda)]}{1 + 3m\phi_I^2 + \lambda(2m^2\phi_I^2 a_{II}) + \left[m^2\phi_I^2\left(\sum_{i=1}^{n} m_{ii} a_{iI}^2\right) + m^3 a_{II}^2 \phi_I\right]\lambda^2} \tag{49}$$

## Special Case: Stiffness Modification Only

Assume a modification of stiffness has a negligible affect on the system mass property, i.e., $[\Delta m] = [0]$. Then

$$\{g\} = \frac{1}{\lambda}\{\phi\} - [Y][\hat{\Delta} k]([I] + [\hat{Y}][\hat{\Delta} k])^{-1}\frac{1}{\lambda}\{\hat{\phi}\} \tag{50}$$

and

$$\{f\}^T\{g\} = \frac{1}{\lambda}\left[1 - \frac{1}{\lambda}\{\hat{\phi}\}^T[\hat{\Delta} k](I + \hat{Y}\hat{\Delta} k)^{-1}\{\hat{\phi}\}\right] \tag{51}$$

Rayleigh's quotient is

$$\lambda_R = \lambda + \{\hat{\phi}\}^T[\hat{\Delta} k]\{\hat{\phi}\} \tag{52}$$

Timoshenko's quotient is

$$\lambda_T = \frac{\{\phi\}[m]\{\phi\}}{\{f\}^T\{g\}} = \frac{\lambda}{[1 - 1/\lambda\{\hat{\phi}\}^T[\hat{\Delta} k]([I] + [\hat{Y}][\hat{\Delta} k])^{-1}\{\hat{\phi}\}]} \tag{53}$$

Let

$$\{g\} = (1/\lambda)(\{\phi\} - [Y][H]\{\hat{\phi}\}) \tag{54}$$

where

$$[H] = [\hat{\Delta} k]([I] + [\hat{Y}][\hat{\Delta} k])^{-1}$$

Then

$$\{g\}^T[m] = (1/\lambda)(\{\phi\}^T[m] - \{\hat{\phi}\}[H]^T[Y]^T[m])$$

Hence,

$$\begin{aligned}\{g\}^T[m]\{g\} = (1/\lambda^2)[&\{\phi\}^T[m]\{\phi\} \\ &- \{\hat{\phi}\}^T[m][Y][H]\{\hat{\phi}\} - \{\hat{\phi}\}^T[H]^T[Y]^T[m]\{\phi\}^T \\ &+ \{\hat{\phi}\}^T[H]^T[Y]^T[m][Y][H]\{\hat{\phi}\}]\end{aligned} \tag{55}$$

Since

$$\{\phi\}^T[m]\{\phi\} = 1.0$$

and

$$\{\phi\}^T[m][Y] = \left(\frac{1}{\lambda}[k]\{\phi\}\right)^T([K]^{-1}[B]^T) = \frac{1}{\lambda}\{\phi\}^T[B]^T = \frac{1}{\lambda}\{\hat{\phi}\}^T$$

$$[Y]^T[m]\{\phi\} = (\{\phi\}^T[m][Y])^T = \frac{1}{\lambda}\{\hat{\phi}\}$$

Thus,

$$\{g\}^T[m]\{g\} = \frac{1}{\lambda^2}[1 - 2\{\hat{\phi}\}^T[H]\{\hat{\phi}\} + \{\hat{\phi}\}^T[H]^T[Y]^T[m][Y][H]\{\hat{\phi}\}] \tag{56}$$

and use Eqs. (53) and (56) to obatin

$$\lambda^{(1)} = \frac{\lambda[1 - A/\lambda]}{1 - 2A + C} \tag{57}$$

where

$$A = (1/\lambda)\{\hat{\phi}\}^T[H]\{\hat{\phi}\} \tag{58}$$

$$C = \{\hat{\phi}\}^T[H]^T[Y]^T[m][Y][H]\{\hat{\phi}\} \tag{59}$$

and

$$[H] = [\hat{\Delta} k]([I] + [\hat{Y}][\hat{\Delta} k])^{-1} \tag{60}$$

This general formulation for stiffness cases modification can be specialized for such cases as placing a spring $k$ between dof $I$ and the ground, introducing a spring between dof $I$ and dof $J$, and modifying the stiffness (cross-sectional) of a space truss element. These are summarized in Table 1.

## Limiting Behavior of Stiffness Elements

The above $\lambda_T$ and $\lambda^{(1)}$ equations for stiffness modifications only are useful in studying the limiting behavior (of $\lambda$) for very stiff added springs. These results are summarized in Table 2. Note that Rayleigh's quotient fails to provide any information regarding the limiting behavior of the stiffness members.

## Numerical Examples

### Example 1: Two dof Spring-Mass System

For the two degree-of-freedom system shown in Fig. 1, the modal data are

$$\lambda_1 = 0.382 \qquad \{\phi_1\} = \begin{Bmatrix} 0.5257 \\ 0.8507 \end{Bmatrix}$$

and

$$\lambda_2 = 2.618 \qquad \{\phi_2\} = \begin{Bmatrix} 0.80 \\ -0.52 \end{Bmatrix}$$

Note that

$$\{\phi_1\}^T[m]\{\phi_1\} = \{\phi_2\}^T[m]\{\phi_2\} = 1.0$$

For this system,

$$[k] = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \text{ and } [a] = [k]^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

Several cases of modifications will be studied.

Case 1: Add spring $k$ between dof 1 and the ground Rayleigh's quotient,

$$\lambda_R = 0.382 + 0.2764K \tag{61}$$

Timoshenko's quotient is

$$\lambda_T = \frac{0.382}{1 - 0.723k/(1+k)} \tag{62}$$

Observe that as $k \to \infty$,

$$\lim_{k \to \infty} \lambda_T = 1.379 \tag{63}$$

**Table 1 Approximate formulas for the mode shape and natural frequency for locally modified systems**

| Modification | Approximate mode shape, $\{g\}$ | Approximate eigenvalue Timoshenko's quotient, $\lambda_T$ | Approximate eigenvalue inverse iteration, $\lambda^{(1)}$ |
|---|---|---|---|
| Add mass $m$ at dof $I$ | $\frac{1}{\lambda}\{\phi\}+\{a_I\}m\phi_I$ | $\frac{\lambda(1+\phi_I^2 m)}{1+2\phi_I^2 m+(\phi_I^2 a_{II} m^2\lambda)}$ | $\lambda[1+2\phi_I^2 m+(\phi_I^2 a_{II} m^2\lambda)]$ $\div\left[(1+3m\phi_I^2)+\lambda(2m^2\phi_I^2 a_{II}) + \left(m^2\phi_I^2\left\{\sum_{i=1}^{n} m_{ii}a_{iI}^2\right\}+m^3 a_{II}^2\phi_I^2\right)\lambda^2\right]$ |
| Add spring $k$ between dof $I$ and ground | $\frac{1}{\lambda}\{\phi\}-\{a_I\}\alpha$ <br> $\alpha=\frac{k\phi_I}{(1+a_{II}k)\lambda}$ | $\frac{\lambda}{1-\alpha\phi_I}$ | $\frac{\lambda(1-\alpha\phi_I)}{1-2\alpha\phi_I+\lambda^2\alpha^2 C}$ <br> $C=\sum_{i=1}^{n} m_{ii}a_{iI}^2$ |
| Add spring $k$ between dof $I$ and $J$ | $\frac{1}{\lambda}\{\phi\}-\{z\}A$ <br> $A=\frac{(\phi_I-\phi_J)k}{\lambda(1+k\bar{a})}$ <br> $\bar{a}=a_{II}-2a_{IJ}+a_{JJ}$ | $\frac{\lambda}{1-A(\phi_I-\phi_J)}$ <br> $\{z\}=\{a_I\}-\{a_J\}$ | $\frac{\lambda(1-A(\phi_I-\phi_J))}{1-2A(\phi_I-\phi_J)+\lambda^2 A^2 C}$ <br> $C=\sum_{i=1}^{n} m_{ii}z_i^2$ |
| Change truss stiffness by $k=\Delta AE/L$ | $\frac{1}{\lambda}\{\phi\}-[Y]\{b\}kh$ <br> $h=\bar{b}\phi/(\lambda(1+k\bar{a}))$ <br> $\bar{b}\phi=\{b\}^T\{\hat{\phi}\}$ <br> $\bar{a}=\{b\}^T[\hat{Y}]\{b\}$ | $\frac{\lambda}{(1-(hk)\bar{b}_\phi)}$ <br> $\{b\}^T=[C_x C_y C_z - C_x - C_y - C_z]$ | $\frac{\lambda(1-(hk)\bar{b}_\phi)}{1-2kh\bar{b}_\phi+\lambda^2(kh)^2\sum_{i=1}^{n} m_{ii}z_i^2}$ <br> $\{z\}=[Y]\{b\}$ |

**Table 2 Limiting case for local stiffness modification ($k\to\infty$)**

| Modification | Limiting value of eigenvalue: Timoshenko's quotient, $\lambda_T$ | Limiting value of eigenvalue: Inverse iteration, $\lambda^{(1)}$ |
|---|---|---|
| Add spring $k$ between dof $I$ and ground | $\frac{\lambda}{1-A}$ <br> $A=\phi_I^2/(\lambda a_{II})$ | $\frac{\lambda(1-A)}{1-2A+BC}$ <br> $B=\phi_I^2/(a_{II})^2$ <br> $C=\sum_{i=1}^{n} m_{ii}a_{iI}$ |
| Add spring $k$ between dof $I$ and dof $J$ | $\frac{\lambda}{1-A(\phi_I+\phi_J)}$ <br> $A=(\phi_I-\phi_J)/(\bar{a}\lambda)$ | $\frac{\lambda(1-B)}{(1-2B)+A^2\lambda^2 C}$ <br> $B=(\phi_I-\phi_J)A$ <br> $C=\sum_{i=1}^{n} m_{ii}(a_{iI}-a_{iJ})$ <br> $\bar{a}=a_{II}-2a_{IJ}+a_{JJ}$ |
| Change truss stiffness by $k=\Delta AE/L$ | $\frac{\lambda}{1-A}$ <br> $A=\bar{b}_\phi^2/(\bar{a}\lambda)$ <br> $\bar{b}_\phi$, $\bar{a}$, $z_i$ are defined in Table 1 | $\frac{\lambda(1-A)}{1-2A+\frac{(\lambda^2)(A^2)}{\bar{b}_\phi^2}(\Sigma m_{ii}z_i^2)}$ |

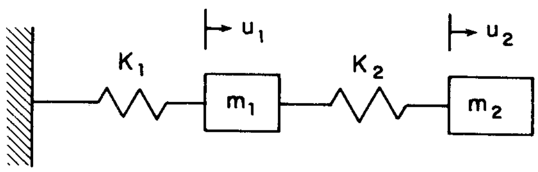


**Fig. 1 Two dof spring-mass system.**

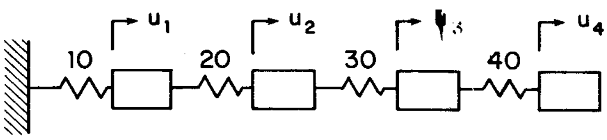


**Fig. 2 Four dof spring-mass system.**

**Table 3 Add spring $k$ between dof 1 and ground of 2 dof system**

| $K$ | $\lambda'$ (exact solution) | $\lambda_R$(error) | $\lambda_T$(error) | $\lambda^{(1)}$(error) |
|---|---|---|---|---|
| 2 | 0.6972 | 0.9347 (+35%) | 0.7379 (+7%) | 0.7037 (0.93) |
| ∞ | 1.0 | ∞ | 1.379 (34%) | 0.990 (−1%) |

The results of using Eqs. (61-63) as well as those of inverse iteration are evaluated with $k=2$ and ∞. They are tabulated in Table 3.

Case 2: Add mass $m$ at dof 2, then Rayleigh's quotient is

$$\lambda_R = \frac{0.382}{1+0.724m} \tag{64}$$

Timoshenko's quotient is

$$\lambda_T = \frac{0.382(1+0.724m)}{1+1.447m+0.553m^2} \tag{65}$$

The inverse iteration is

$$\lambda^{(1)} = \frac{(1+1.447m+0.553m^2)\lambda}{[(1+2.171m)+(2.8948m^2)\lambda]}$$

$$+[3.618m^2\lambda^2+(2.8948)m^3\lambda^2] \tag{66}$$

where $\lambda = 0.382$.

The results for $m=1$ and 10 are provided in Table 4.

Case 3: Add spring $k$ between dof 1 and 2. Rayleigh's quotient is then

$$\lambda_R = 0.382 + 0.10562k \tag{67}$$

Timoshenko's quotient is

$$\lambda_T = \frac{0.382}{1-0.2765k/(1+k)} \tag{68}$$

$$\lim_{k\to\infty} \lambda_T = \frac{0.382}{1-0.2765} = 0.528 \tag{69}$$

For inverse iteration, the limiting case is

$$\lim_{k\to\infty} \lambda''' = 0.5001$$

The results for $k=1$, 10, and ∞ are given in Table 5.

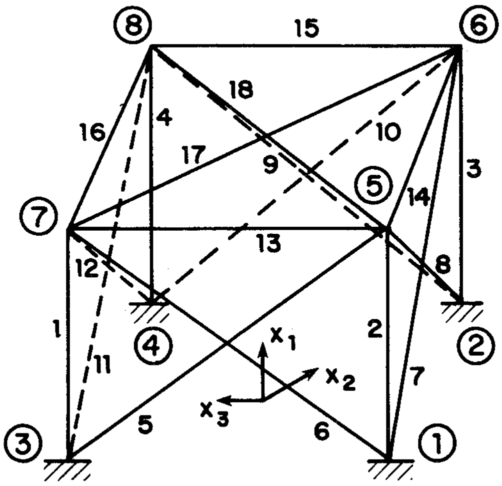


**Fig. 3 Twelve dof space truss model.**

**Table 4 Add spring $k$ between dof 1 and dof 2**

| $K$ | $\lambda'$ | $\lambda_R$(error) | $\lambda_T$(error) | $\lambda^{(1)}$(error) |
|---|---|---|---|---|
| 1 | 0.438447 | 0.4876 (11%) | 0.4433 (1.1%) | |
| 10 | 0.48864 | 1.4383 (1.4%) | 0.5103 (4.4) | 0.4893 (0.14%) |
| ∞ | 0.5 | ∞ | 0.528 (5.6%) | 0.5001 (0.03%) |

**Table 5 Add mass $m$ at dof 2**

| $K$ | $\lambda'$ | $\lambda_R$(error) | $\lambda_T$(error) | $\lambda^{(1)}$(error) |
|---|---|---|---|---|
| 1 | 0.219223 | 0.2215 (1.07%) | 0.219522 (0.14%) | 0.21924 (0.008%) |
| 10 | 0.44422 | 0.46 (4.5%) | 0.444776 (0.13%) | 0.44426 (0.009%) |

**Table 6 Add spring $k$ between dof 1 and ground for the 4 dof system of Fig. 2**

| $K$ | $\lambda'$ | $\lambda_R$(error) | $\lambda_T$(error) | $\lambda^{(1)}$(error) |
|---|---|---|---|---|
| 5 | 0.661 | 0.679 (2.72%) | 0.664 (0.45%) | 0.66124 (0.08%) |
| 140 | 1.596 | 6.102 (382%) | 2.2127 (38.6%) | 1.664 (4.3%) |

**Example 2: Four dof Spring-Mass System**

The system and data are shown in Fig. 2. For Case 1, add spring $k$ between dof 1 and the ground. The results are given in Table 6 for $k=5$ and 140.

**Example 3: Twelve dof Space Truss Model of Helicopter Tail-Beam Structure**

Figure 3 shows an 8 node, 18 member, 12 dof space truss model. This model is one bay of a six-bay tail boom truss structure of a helicopter.[31] In the original model, we assume all members have a cross-sectional area of 0.2 in.$^2$ and are made of aluminum with $E=10^7$ psi. The mass (including nonstructured mass) is assumed to be 0.1 lb·s$^2$/in.$^2$. Table 7 shows the results of adding masses or springs as well as modifying the truss member stiffnesses. From the results, it is observed that $\lambda^{(1)}$ (inverse iteration) yields excellent results even when the magnitude of the modification is large.

Table 7 Twelve dof space truss results

| Case | Approximate eigenvalue $\lambda_R$ | $\lambda_T$ | $\lambda^{(1)}$ | Exact solution $\lambda'$ |
|---|---|---|---|---|
| Add mass (1.0) at dof 1 | 6350.8 | 6349.5 | 6349.3 (0.02%)[a] | 6347.8 |
| Add $m=10$ at dof 1 | 6140.6 | 6205.9 | 5955.1 (4.97%)[a] | 5673 |
| Add $m=1$ at dof 2 | 5649 | 5610 | 5591 (0.57%)[a] | 5558.69 |
| Add $k=1000$ between dof 2 and ground | 6504 | 6502 | 6501 | 6455.94 |
| $\Delta Aq=0.02$ in.$^2$ | | | 6453.5 (1.5%)[a] | 6405.6 |
| $\Delta Aq=2$ in.$^2$ | | | 6505.6 (1.5%)[a] | 6411.8 |
| $\Delta Aq=20$ in.$^2$ | | | 6510 (0.8%)[a] | 6412.6 |

[a]error $=\lambda^{(1)}-\lambda'/\lambda'\times 100$.

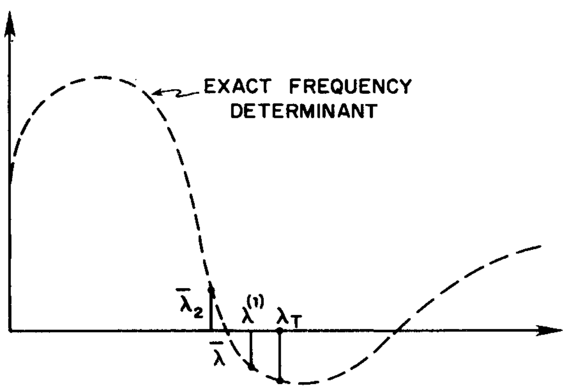


Fig. 4 Improved root searching scheme.

## Conclusions and Recommendations

The reanalysis equations derived in this paper are simple algebraic equations. For reanalysis, these relations can be used to compute readily an accurate approximate value of the modified system. If more accurate solutions are needed, the present approach can be combined with previous nonlinear eigenvalue problem reanalysis formulations. This is illustrated in Fig. 1. That is, compute $\lambda_T$, $\lambda_1$, and $\lambda_2$ and then $\lambda_2<\lambda'<\lambda^{(1)}$, using a three-point root searching algorithm (Muller's method) to compute the improved eigenvalue $\bar{\lambda}$. (See Fig. 4.)

In synthesis problems using the $\lambda^{(1)}$ formulation, a more accurate parameterization of the eigenvalue as a function of design parameters can be obtained. For some cases, these can be represented as polynomials in the design parameter. More accurate linearization methods may be realizable from this equation.

The present method works only for the fundamental mode. Extension of this approach to higher modes should be attempted. Perhaps some form of Rayleigh's quotient iteration[32] will be successful.

Finally, the present method is suitable for buckling load reanalysis, since in buckling problems only the lowest eigenvalues are usually of concern.

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